The uniform controllability property of semidiscrete approximations for the parabolic distributed parameter systems in Banach spaces

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Abstract

The problem we consider in this work is to minimize the L^q-norm ($q > 2$) of the semidiscrete controls. As shown in [LT06], under the main approximation assumptions that the discretized semigroup is uniformly analytic and that the degree of unboundedness of control operator is lower than 1/2, the uniform controllability property of semidiscrete approximations for the parabolic systems is achieved in L^2. In the present paper, we show that the uniform controllability property still continue to be asserted in $L^q (q > 2)$ even with the condition that the degree of unboundedness of control operator is greater than 1/2. Moreover, the minimization procedure to compute the approximation controls is provided. An example of application is implemented for the one dimensional heat equation with Dirichlet boundary control.

1 Introduction

Consider an infinite dimensional linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \qquad y(0) = y_0, \tag{1}$$

where the state y(t) belongs to a reflexive Banach space X, the control u(t) belongs to a reflexive Banach space U, $A : D(A) \to X$ is an operator, and

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B is a control operator (in general, unbounded) on U. Discretizing this partial differential equation by using, for instance, a finite difference or a finite element scheme, leads to a family of finite dimensional linear control systems

$$\dot{y_h}(t) = A_h y_h(t) + B_h u_h(t), \qquad y_h(0) = y_{0h}, \tag{2}$$

where $y_h(t) \in X_h$ and $u_h(t) \in U_h$, for $0 < h < h_0$.

Let $y_1 \in X$. If the control system (1) is controllable in time T then there exists a solution $y(.)$ of (1) associated with a control u such that $y(T) = y_1$. As known, we have many available methods in order to identify the controllability. We refer to J.-L. Lions [L88] for a well-known method in attainning the control of the minimal L^2 norm- the so called HUM (HUM stands for Hilbert uniqueness method). In this work, however, we investigate a method which we can achieve the minimization procedure in L^q norm $(q > 1)$. Namely, we will establish some conditions ensuring the existence and convergence of the discretized control of the minimal L^q norm

$$\min \frac{1}{q} \int_0^T \|u_h(t)\|^q dt \quad (q > 1). \tag{3}$$

Necessary conditions for optimal control in finite dimensional state space were derived by Pontryagin et al [PS62] (see also [E], [T05]). The Maximum Principle as a set of necessary conditions for optimal control in infinite dimensional space has been studied by many authors. Since it is well known that the Maximum Principle may be false in infinite dimensional space, there are still many papers that give some conditions to ensure that the Maximum Principle remains true. It was Li and Yao [LY85] who used the Eidelheit separation theorem and of the Uhl's theorem in order to extend the Maximum Principle to a large class of problems in infinite dimensional spaces when the target set is convex and the final time T is fixed. Additionally, the authors of [F87], [FF91], [LY91], by making use of Ekeland's variational principle, give some conditions on the reachable set and on the target set in order to get an extension of Maximum Principle. Nevertheless, the problem is that when we applied the result of [F87], [LY91] for the system (1) in the case the final state and final time are fixed, the finite-codimensional condition in [F87], [LY91] does not satisfy for the system (1) in general. Hence we cannot use Maximum Principle in our problem. Fortunately, thank to the Fenchel-Rockafellar duality theorem which is used in the same manner in [CGL94], [GLH08], the constrainted minimization of the function can be replaced by the unconstrainted minimization problem of corresponding conjugate function. Therefore, we will consider the above minimization procedure in the same framework with [GLH08].

The uniform controllability is an important area of control theory research and it has been the subject of many papers in recent years. The main goal of this article is to establish conditions ensuring a uniform controllability property of the family of discretized control systems (2) in L^q and to establish a computationally feasible approximation method for identifying controllability.

It is well known that controllabilty and observability are dual aspects of the same problem. We therefore will focus on the uniform observability which is shown to hold when the observability constant of the finite dimensional approximation systems does not depend on h. Some relevant references concerning this property has been investigated by many authors in series of articles [IZ99], [LZ98], [LZ02], [NZ03], [Zua99], [Zua02], [Zua04], [Zua05], [Zua06], [BHL10a], [BHL10b] and [LT06]. For finite difference schemes, a uniform observability property holds for one-dimensional heat equation [LZ02], beam equation [LZ98], Schrodinger equations [Zua05], but does not hold for 1-D wave equations [IZ99]. This is due to the fact that the discrete dynamics generates high frequency spurious solutions for which the group velocity vanishes that do not exist at the continuous level. To overcome these high frequency spurious for wave equations, [Zua05] showed some remedies such that Tychonoff's regularization, multigrid method, mixed finite element and filtering of high frequency, etc.

To our knowledge, in 1-D heat equation case, due to the fact that the dissipative effect of the 1-D heat equation acts as a filtering mechanicsm by itself and it is strong enough to exclude high frequency spurious oscillations[LZ98]. However, the situation is more complex in multi-dimensional. The counterexample is shown in [Zua06] for the simplest finite difference semi-discretization scheme for the heat equation in the square.

In recent works in L^2-norm, by means of discrete Carleman inequalities, the authors in [BHL10a], [BHL10b] obtain the weak uniform observability inequality for parabolic case by adding reminder terms of the form $e^{-Ch^{-2}} \|\psi_h(T)\|^2_{L^2(\Omega)}$ which vanishes asymptotically as $h \to 0$. Moreover, as in [LT06], the approximate controllability is derived from using semigroup arguments and introducing a vanishing term of the form $h^\beta \|\psi_h(T)\|^2_{L^2(\Omega)}$ for some $\beta > 0$.

In fact, an efficient computing the null control for a numerical approximation scheme of the heat equation is itself a difficult problem. Accoding to the pioneering work of Carthel, Glowinski and Lions in [CGL94], the null control problem is reduced to the minimization of a dual conjugate function with respect to final condition of the adjoint state. However, as a consequence of high regularizing property of the heat kenel, this final condition does not belong to L^2, but a much large space that can hardly be approximated by

standard techniques in numerical analysic. Recently, A. Munch and collaborators have developed some feasible numericals such that the transmutation method, variational approach, dual and primal algorithms allow to more efficiently compute the null control (see in series [CM09], [CM10], [MZ10], [PM10]).

The discretization framework in this paper is the same spirit as [LT06], [LT00]. In [LT06], under standard assumptions on the discretization process and for an exactly null controllable parabolic system (1), if **the degree of unboundedness of the control operator is lower than 1/2** then the semidiscrete approximation models are uniformly controllable and they also showed that for the (2), the minimizing of the cost function of discretized control with power $q = 2$ is obtained.

In this article, we prove the existence of the minimum of the cost function of discretized control power q ($q > 2$) for type (2), in the case the operator A generates an analytic semigroup. Of course, due to regularization properties, the control system (1) is not exactly controllable in general. Hence, we focus on exact null controllability. Our main result, theorem 3.1, states that for exactly null controllable parabolic system (1) and under standard approximation assumptions, if the discretized semigroup is uniformly analytic, and if **the degree of unboundedness of the control operator B with respect to A is greater than 1/2**, then a uniform observability inequality in (L^p) is proved. We stress that we do not prove uniform exact null controllability property for the approximating system (2). Moreover, a minimization procedure to compute the approximation controls is provided.

The outline of the paper is as follows. In Section 2, we briefly review some well-known facts on controllability of linear partial differential equation in reflexive Banach spaces. In Section 3, we consider the existence and unique solution of the minimization problem in continuous case. By making use of the Fenchel-Rockafellar duality theorem, we gives a constructive way to build the control of minimal L^q norm. The main result is stated in Section 4 and proved in Section 5. An example of application and numerical simulations are provided in Section 6, for the one-dimensional heat equation with Dirichlet boundary control. An Appendix is devoted to the proof of a lemma.

2 A short review on controllability of linear partial differential equations in reflexive Banach spaces

In this section, it is convenient to first have a quick look to controllability of infinite dimensional linear control systems in reflexive Banach spaces (see more [CT06], [P83], [TW07]).

The notation $L(E,F)$ stands for the set of linear continuous mappings from E to F, where E and F are reflexive Banach spaces.

Let X be a reflexive Banach space. Denote by $<,>_X$ the inner product on X, and by $\|.\|_X$ the associated norm. Let $S(t)$ denote a strongly continuous semigroup on X, of generator $(A, D(A))$. Let X_{-1} denote the completion of X for norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$, where $\beta \in \rho(A)$ is fixed. Note that X_{-1} does not depend on the specific value of $\beta \in \rho(A)$. The space X_{-1} is isomorphic to $(D(A^*))'$, the dual space of $D(A^*)$ with respect to the pivot space X, and $X \subset X_{-1}$, with a continuous and dense embedding. The semigroup $S(t)$ extends to a semigroup on X_{-1}, still denoted $S(t)$, whose generator is an extension of the operator A, still denoted A. With these notations, A is a linear operator from X to X_{-1}.

Let U be a reflexive Banach space. Denote by $<,>_U$ the inner product on U, and by $\|.\|_U$ the associated norm.

A linear continuous operator $B : U \to X_{-1}$ is admissible for the semigroup $S(t)$ if every solution of

$$y' = Ay(t) + Bu(t), \tag{4}$$

with $y(0) = y_0 \in X$ and $u(.) \in L^q(0, +\infty; U)$, satisfies $y(t) \in X$, for every $t \geq 0$. The solution of equation (1) is understood in the mild sense, i.e,

$$y(t) = S(t)y(0) + \int_0^T S(t-s)Bu(s)ds, \tag{5}$$

for every $t \geq 0$.

For $T > 0$, define $L_T : L^q(0,T;U) \to X_{-1}$ by

$$L_T u = \int_0^T S(T-s)Bu(s)ds. \tag{6}$$

A control operator $B \in L(U, X_{-1})$ is admissible, if and only if $Im L_T \subset X$, for some (and hence for every) $T > 0$.

The adjoint L_T^* of L_T satisfies

$$\begin{aligned} &L_T^* : X^* \to (L^q(0,T;U))^* = L^p(0,T;U^*) \\ &L_T^*\psi(t) = B^*S(T-t)^*\psi \end{aligned} \tag{7}$$

, a.e on [0,T] for every $\psi \in D(A^*)$. Moreover, we have

$$\|L_T^*\psi\| = \sup_{\|u\|_q \leq 1} \int_0^T < B^*S(T-s)^*\psi, u(s) > ds, \tag{8}$$

for every $\psi \in X^*$.

Let $B \in L(U, X_{-1})$ denote an admissible control operator.

We use two following lemmas (for the proofs we refer to [TW07])

Lemma 1. *Z, X are reflexive Banach spaces. $G \in L(Z, X)$ then the following statements are equivalent:*

- *G is onto.*
- *G^* bounded from below i.e there exists $C > 0$ such that*

$$\|G^*x\|_Z \geq C\|x\|_X \ \ \textit{every } x \in X.$$

Lemma 2. *Z_1, Z_2, Z_3 are reflexive Banach spaces. And $f \in L(Z_1, Z_3)$, $g \in L(Z_2, Z_3)$. Then the following statements are equivalent:*

- *$Imf \subset Img$.*
- *There exists a constant $C > 0$ such that : $\|f^*z\|_{Z_1} \leq C\|g^*z\|_{Z_2}$ for every $z \in Z_3$.*
- *There exists an operator $h \in L(Z_1, Z_2)$ such that $f = gh$.*

We state some concepts as follows

For $y_0 \in X$, and $T > 0$, the system (4) is exactly controllable from y_0 in time T if for every $y_1 \in X$, there exists $u(.) \in L^q(0,T;U)$ so that the corresponding solution (4), with $y(0) = y_0$ satisfies $y(T) = y_1$.

In fact that the system (4) is exactly controllable from y_0 in time T if and only if L_T is onto, that is $ImL_T = X$. Making use of Lemma 1, there exists $C > 0$ such that

$$\begin{aligned} C\|\psi\|_X \leq \|L_T^*\psi\| &= \sup_{\|u\|_q \leq 1} \int_0^T < B^*S(T-s)^*\psi, u(s) > ds \\ &\leq \sup_{\|u\|_q \leq 1} \int_0^T \|B^*S(T-t)^*\psi\|\|u(t)\|dt \\ &\leq (\int_0^T \|B^*S(t)^*\psi\|^p dt)^{\frac{1}{p}}. \end{aligned}$$

Therefore, the system (4) is exactly controllable in time T if and only if

$$\int_0^T \|B^*S(t)^*\psi\|^p dt \geq C\,\|\psi\|_X^p\,. \tag{9}$$

For $T > 0$, the system (4) is said to be exactly null controllable in time T if for every $y_0 \in X$, there exists $u(.) \in L^q(0,T;U)$ so that the corresponding solution of (4), with $y(0) = y_0$ satisfies $y(T) = 0$.

This means that the system (4) is exactly null controllable in time T if and only if $ImS(T) \subset ImL_T$. Making use of Lemma 2 and the same argument as above, there exists $C > 0$ such that

$$C\,\|S(T)^*\psi\|_X \leq \|L_T^*\psi\| \leq (\int_0^T \|B^*S(t)^*\psi\|^p dt)^{\frac{1}{p}}.$$

Thus, the system (4) is exactly null controllable in time T if and only if

$$\int_0^T \|B^*S(t)^*\psi\|^p dt \geq C\,\|S(T)^*\psi\|_X^p\,. \tag{10}$$

3 Duality

The goal of this section is to show that, with using duality arguments and Fenchel- Rockafellar theorem we can achieve the control of minimal L^q norm $(q > 1)$for the continuous framework.

Consider the system :

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \text{ on } Q_T = (0,T)\times\Omega \\ y(0) = y_0 \end{cases} \tag{11}$$

where B is admissible and A generates an analytic semigroup $S(t)$ in the reflexive Banach space X.

Our aim is to mimimize the following functional:

$$\begin{cases} \text{Minimize } J(u) = \frac{1}{q}\int_0^T \|u\|^q\,dt \quad (q > 1) \\ \text{Subject to } u \in E \end{cases} \tag{12}$$

where $E = \{u \in U :$ u steering the system from y_0 at time zero to y(T)=0$\}$.

Theorem 1. *The problem (12) has a unique solution u.*

Proof. First of all, we show the existence of the solution of the optimal problem.

Consider a minimizing sequence $(u_n)_{n\in N}$ of controls on $[0, T]$, i.e,

$$\int_0^T \|u_n\|^q \, dt \text{ converges to } \inf J(u) \text{ as } n \to +\infty. \tag{13}$$

Hence, $(u_n)_{n\in N}$ bounded in $L^q(0, T; U)$.

Since U is reflexive space and $q < +\infty$, then $L^q(0, T; U)$ is also reflexive.

Thus, up to a sequence, u_n converges weakly to u in L^q. Note that the trajectory y_n (resp. y) associated with the control u_n (resp. u) on [0,T] through the system

$$\dot{y}_n = Ay_n + Bu_n, \; y_n(0) = y_0,$$

and the solution of the above system is expressed in form

$$y_n(t) = S(t)y_0 + \int_0^T S(T - s)Bu_n ds.$$

A passage to the limit imples that

$$\dot{y} = Ay + Bu, \; y(0) = y_0,$$

and the solution y associated with control u in the form

$$y(t) = S(t)y_0 + \int_0^T S(T - s)Buds.$$

As u_n converges weakly to u in L^q, we get the inequality

$$\begin{aligned} \int_0^T \|u(t)\|^q \, dt &\leq \liminf_{n\to+\infty} \int_0^T \|u_n(t)\|^q \, dt \\ &= \inf \int_0^T \|u(t)\|^q \, dt. \end{aligned}$$

It follows easily that

$$\int_0^T \|u(t)\|^q \, dt = \inf \int_0^T \|u(t)\|^q \, dt.$$

Hence u is optimal of (12). This ensures the existence of a optimal control.

Moreover, the cost function is strictly convex then the solution is obvious uniqueness. □

By making use of convex duality, the problem of control to trajectories is reduced to the minimization of the corresponding conjugate function. Roughly speaking, it is stated through the following theorem:

Theorem 2. *(i) We have the identity:*

$$\inf_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\,dt = -\inf_{\psi_T}\Big(\frac{1}{p}\int_0^T \|B^*\psi\|^p\,dt + <\psi(0), y_0>\Big), \tag{14}$$

where ψ be solution of :

$$-\dot{\psi} = A^*\psi \tag{15}$$

$$\psi(T) = \psi_T. \tag{16}$$

Or, we have in the form

$$\inf_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\,dt = -\inf_{\psi\in X^*}\Big(\frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\,dt + <S(T)^*\psi, y_0>\Big). \tag{17}$$

(ii) If u_{op} is optimal of the problem (12) then

$$u_{op}(t) = \|B^*S(T-t)^*\varphi\|^{p-2}\,B^*S(T-t)^*\varphi,$$

where φ be optimal of the function:

$$J^*(\psi) = \frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\,dt + <S(T)^*\psi, y_0> .$$

Proof.
(i) Let $\bar{y}$ be solution of (1) with $u = 0$ and we introduce the operator $N \in L(L^q(Q_T), X)$ with $Nu = z_u(.,T)$ for all $u \in L^q(Q_T)$, where z_u is solution to

$$\dot{z} = Az + Bu \tag{18}$$

$$z(x,0) = 0. \tag{19}$$

Accordingly, the solution y of (11) can be decomposed in the form

$$y = z_u + \bar{y}. \tag{20}$$

The adjoint N^* is given as follows
For each $\psi_T \in X^*$, $N^*\psi_T = B^*\psi$ where ψ is solution of (15), (16).
Let us introduce the following functions F and G

$$F(z_T) = \begin{cases} 0 & \text{for } z(T) = -\bar{y}(T) \\ +\infty & \text{otherwise} \end{cases} \quad ,$$

$$G(u) = \frac{1}{q} \int_0^T \|u\|^q \, dt.$$

Then, the problem (12), where the infimium is taken over all u satisfying E, is equivalent to the following minimization problem

$$\inf_{u \in L^q(Q_T)} (F(Nu) + G(u)). \tag{21}$$

We can apply now duality theorem of W.Fenchel and T.R.Rockafellar (see Theorem 4.2 p.60 in [ET99]). It gives

$$\inf_{u \in L^q(Q_T)} (F(Nu) + G(u)) = - \inf_{\psi_T \in X^*} (G^*(N^*\psi_T) + F^*(-\psi_T)), \tag{22}$$

where F^* and G^* are the convex conjugate of F and G, respectively. Denote that $\psi_T = \psi(T)$, $z_T = z(T)$.

Note that

$$F^*(\psi_T) = \sup_{z_T = -\bar{y_T}} < z_T, \psi_T >= - < \psi_T, \bar{y_T} >,$$

for all $\psi_T \in X^*$.

Additionally,

$$G^*(\omega) = \frac{1}{p} \int_0^T \|\omega\|^p \, dt.$$

Therefore,

$$G^*(N^*\psi_T) + F^*(-\psi_T) = \frac{1}{p} \int_0^T \|B^*\psi\|^p \, dt + < \psi_T(x), \bar{y_T}(x) > . \tag{23}$$

Finally, multiplying the state equation (15) by $\bar{y}$ and due to (11), we obtain

$$< \psi_T, \bar{y_T} >=< \psi(0), y_0 > .$$

Rewrite (23) as follows

$$\begin{aligned} G^*(N^*\psi_T) + F^*(-\psi_T) &= \frac{1}{p} \int_0^T \|B^*\psi\|^p \, dt + < \psi(0), y_0 > \\ &= \frac{1}{p} \int_0^T \|B^* S(T-t)^* \psi_T\|^p \, dt + < S(T)^* \psi_T, y_0 >, \end{aligned}$$

since ψ is the solution of (15) , (16).

From (21) and (22), we have the identity

$$\inf_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\,dt = -\inf_{\psi_T}(\frac{1}{p}\int_0^T \|B^*\psi\|^p\,dt + <\psi(0), y_0>),$$

where ψ be solution of (15), (16).

Or, we have in the form

$$\inf_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\,dt = -\inf_{\psi\in X^*}(\frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\,dt + <S(T)^*\psi, y_0>).$$

(ii) If we denote by (u_{op}) , (φ_T) the unique solution to "LHS of (14)" and "RHS of (14)" respectively, then one has

$$0 = \frac{1}{q}\int_0^T \|u_{op}\|^q\,dt + \frac{1}{p}\int_0^T \|B^*\varphi_T\|^p\,dt + <\varphi_T(0), y_0> . \tag{24}$$

We apply Young's inequality for the first two terms of RHS (24)

$$\frac{1}{q}\int_{Q_T} \|u_{op}\|^q\,dt + \frac{1}{p}\int_{Q_T} \|B^*\varphi_T\|^p\,dt \geq \int_{Q_T} u_{op}.B^*\varphi_T. \tag{25}$$

Then, "RHS of (24)" $\geq \int_{Q_T} u_{op}.B^*\varphi_T + <\varphi_T(0), y_0>$.

Futhermore, by multiplying two sides of (15) by y and applying Green's formula, we obtain

$$<B^*\varphi_T, u> + <\varphi_T(0), y_0> = 0. \tag{26}$$

On the one hand, "RHS of (24)" ≥ 0 (due to (26)).

On the other hand, "RHS of (24)" $= 0$ (due to (24)).

This is equivalent to that the sign "=" in inequality (14) happens, i.e,

$$\|u_{op}\|^q = \|B^*\varphi_T\|^p .$$

It is also rewritten as follows

$$u_{op}(t) = \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi,$$

where φ be optimal of the function J^* is given as above.

$\square$

Remark: It is easily seen that the functional J^* is convex, and from the inequality (10), is coercive. Then, it follows that J^*attains a unique minimum in some point $\varphi \in D(A^*)$. As above explanation, the control $\bar{u}$ is chosen by

$$\bar{u}(t) = \|B^* S(T-t)^* \varphi\|^{p-2} B^* S(T-t)^* \varphi, \tag{27}$$

for every $t \in [0, T]$ and let y(.) be the solution of (11), such that $y(0) = y_0$, associated with the control $\bar{u}$, then we have y(T)=0.

Therefore, $\bar{u}$ is the control of minimal of L^q norm, among all controls whose associated trajectory satisfied $y(T) = 0$.

We emphasize that observability in L^p norm $(1 < p < 2)$ implies controllability and gives a constructive way to build the control of minimal L^q norm $(q > 2)$. A similar result was known in L^2 norm through using HUM (see more [CT06], [L88], [Zua04], [Zua05]).

4 The main result

We are concerned in this work with the uniform controllability property for the parabolic systems. As shown in [LT06], this property is known to hold with the degree of unboundedness of control operator $\gamma \in [0, 1/2)$. In this section, we also establish some appropriate assumptions and conditions ensuring that the unform controllability still holds in the case $\gamma \in \left[1/2, \frac{1}{p}\right)$.

Let X and U be Hilbert spaces, and let $A : D(A) \to X$ be a linear operator and self-adjoint, generating a strongly continuous semigroup $S(t)$ on X. Let $B \in L(U, D(A^*)')$ be a control operator. We make the following assumptions that will be used along this article (also refer to [LT06])

(H1) The semigroup S(t) is analytic.

Therefore, (see [P83]) there exist positive real number C_1 and Ω such that

$$\|S(t)\|_X \leqslant C_1 e^{\omega t} \|y\|_X \,, \|AS(t)y\|_X \leqslant C_1 \frac{e^{\omega t}}{t} \|y\|_X \,, \tag{28}$$

for all $t > 0$ and $y \in D(A)$, and such that, if we set $\hat{A} = A - \omega I$, for $\theta \in [0, 1]$ and there holds

$$\left\|(-\hat{A}^{\theta})S(t)y\right\|_X \leq C_1 \frac{e^{\omega t}}{t^{\theta}} \|y\|_X \,, \tag{29}$$

for all $t > 0$ and $y \in D(A)$.

Of course, inequalities (28) hold as well if one replaces A by A^*, $S(t)$ by $S(t)^*$, for $y \in D(A^*)$.

Moreover, if $\rho(A)$ denotes the resolvent set of A, then there exists $\delta \in \left(0, \frac{\pi}{2}\right)$ such that $\rho(A) \supset \Delta_\delta = \left\{\omega + \rho e^{i\theta} | \theta > 0, |\theta| \leq \frac{\pi}{2} + \delta\right\}$.

For $\lambda \in \rho(A)$, denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent of A . It follows from the previous estimates that exists $C_2 > 0$ such that

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{C_2}{|\lambda - \omega|}, \|AR(\lambda, A)\|_{L(X))} \leq C_2, \tag{30}$$

for every $\lambda \in \Delta_\delta$, and

$$\left\|R(\lambda, \hat{A}))\right\|_{L(X)} \leq \frac{C_2}{|\lambda|}, \left\|\hat{A}R(\lambda, \hat{A})\right\|_{L(X)} \leq C_2, \tag{31}$$

for every $\lambda \in \Delta_\delta + \omega$. Similarly, inequalities (30) and (31) hold as well with A^* and $\hat{A}^*$.

(H2) The degree of unboundedness of B is γ. Assume that $\gamma \in \left[1/2, \frac{1}{p}\right)$ (where p,q are conjugate, i.e $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < 2$). This means that

$$B \in L(U, D((-\hat{A}^*)^\gamma)'). \tag{32}$$

In these conditions, the domain of B^* is $D(B^*) = D((-\hat{A}^*)^\gamma)$, and there exists $C_3 > 0$ such that

$$\|B^*\psi\| \leq C_3 \left\|((-\hat{A}^*)^\gamma)\psi\right\|_X, \tag{33}$$

for every $\psi \in D((-\hat{A}^*)^\gamma)$.

(H3) We consider two families $(X_h)_{0<h<h_0}$ and $(U_h)_{0<h<h_0}$ of finite dimentional spaces, where h is the discretization parameter.

For every $h \in (0, h_0)$, there exist the linear mappings $P_h : D((-\hat{A}^*)^{\frac{1}{2}})' \to X_h$ and $\tilde{P}_h : X_h \to D((-\hat{A}^*)^{\frac{1}{2}})$ and $(\hat{A}^*)^{-\gamma+\frac{1}{2}} : D(-(\hat{A}^*)^{\frac{1}{2}}) \to D(-(\hat{A}^*)^\gamma)$ (resp., there exist linear mappings $Q_h : U \to U_h$ and $\tilde{Q}_h : U_h \to U$), satisfying the following requirements:

$(H_{3.1})$ For every $h \in (0, h_0)$. The following properties hold

$$P_h\tilde{P}_h = id_{X_h} \text{ and } Q_h\tilde{Q}_h = id_{U_h}. \tag{34}$$

$(H_{3.2})$ There exist $s > 0$ and $C_4 > 0$ such that there holds, for every $h \in (0, h_0)$,

$$\left\|\left(I - (\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_hP_h\right)\psi\right\|_X \leq C_4h^s \|A^*\psi\|_X, \tag{35}$$

$$\left\| ((-\hat{A}^*)^\gamma)\left(I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h\right) \psi \right\|_X \leq C_4 h^{s(1-\gamma)} \|A^*\psi\|_X , \qquad (36)$$

for every $\psi \in D(A^*)$ and

$$\left\| (I - \tilde{Q}_h Q_h) u \right\|_U \to 0, \qquad (37)$$

for every $u \in U$, and

$$\left\| (I - \tilde{Q}_h Q_h) B^*\psi \right\|_U \leq C_4 h^{s(1-\gamma)} \|A^*\psi\|_X, \qquad (38)$$

for every $\psi \in D(A^*)$

For every $h \in (0, h_0)$, the vector space X_h (resp. U_h) is endowed with the norm $\|.\|_{X_h}$ (resp. $\|.\|_{U_h}$) defined by:

$\|y_h\|_{X_h} = \left\| \tilde{P}_h y_h \right\|_X$ for $y_h \in X_h$ (resp. ,$\|u_h\|_{U_h} = \left\| \tilde{Q}_h u_h \right\|_U$).

Therefore, we have the properties

$$\left\| \tilde{P}_h \right\|_{L(X_h, X)} = \left\| \tilde{Q}_h \right\|_{L(U_h, U)} = 1 \text{ and } \left\| (\hat{A}^*)^{-\gamma+\frac{1}{2}} x \right\|_X \leq C \|x\|_X , \qquad (39)$$

$$\|P_h\|_{L(X, X_h)} \leq C_5 \text{ and } \|Q_h\|_{L(U, U_h)} \leq C_5. \qquad (40)$$

$(H_{3.3})$ For every $h \in (0, h_0)$, there holds

$$P_h = \tilde{P}_h{}^* \text{ and } Q_h = \tilde{Q}_h{}^*, \qquad (41)$$

where the adjoint operators are considered with respect to the pivot spaces X, U, X_h, U_h.

$(H_{3.4})$ There exists C_6 such that

$$\left\| B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h \psi_h \right\|_U \leq C_6 h^{-\gamma s} \|\psi_h\|_{X_h} , \qquad (42)$$

for all $h \in (0, h_0)$ and $\psi_h \in X_h$.

For every $h \in (0, h_0)$, we define the approximation operators $A_h^* : X_h \to X_h$ of A^* and $B_h^* : X_h \to U_h$ of B^*, by

$$A_h^* = P_h A^* \tilde{P}_h \text{ and } B_h^* = Q_h B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h. \qquad (43)$$

(H_4) The following properties hold:

$(H_{4.1})$ The family of operators e^{tA_h} is uniformly analytic, in sense that there exists $C_7 > 0$ such that

$$\left\| e^{tA_h} \right\|_{L(X_h)} \leq C_7 e^{\omega t}, \qquad (44)$$

$$\left\|A_h e^{tA_h}\right\|_{L(X_h)} \leq C_7 \frac{e^{\omega t}}{t}, \tag{45}$$

for all $t > 0$ and $h \in (0, h_0)$.

($H_{4.2}$) There exists $C_8 > 0$ such that, for every $f \in X$ and every $h \in (0, h_0)$, the respective solutions of $\hat{A}^* \psi = f$ and $\hat{A}_h^* \psi_h = P_h f$ satisfy

$$\|P_h \psi - \psi_h\|_{X_h} \leq C_8 h^s \|f\|_X . \tag{46}$$

In other words, there holds $\left\|P_h \hat{A}^{*-1} - \hat{A}^{*-1} P_h\right\|_{L(X,X_h)} \leq C_8 h^s$.

Remark 4.1 Compare to [LT06], the important point to note here is the appearance of the function $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ in (35), (36) and (42).

According to [LT06], the inequality (22) make sense since $\gamma < \frac{1}{2}$ and thus $im\tilde{P}_h \subset D((-\hat{A}^*)^{1/2}) \subset D((-\hat{A}^*)^{\gamma})$.

In our context, on account of $\gamma \geq \frac{1}{2}$, the inequality (36), which is similar to inequality (22) in [LT06], only make sense if we add the functional $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ in order that $im(\hat{A^*}^{-\gamma+\frac{1}{2}} \tilde{P}_h) \subset D((-\hat{A}^*)^{\gamma})$. The choice of the function $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ seems to be the best adapted to our theory.

Namely, we give here for instance about the functional $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ through the heat equation with Dirichlet boundary control as follows

$$\dot{y} = \Delta y + c^2 y \text{ in } (0,T) \times \Omega$$

$$y(0,.) = y_0 \text{ in } \Omega$$

$$y = u \text{ in } (0,T) \times \Gamma = \Sigma.$$

Set $X = L^2(\Omega)$ and $U = L^2(\Gamma)$. It can be written in the form (4), where the self-adjoint operator $A : D(A) \to X$ is defined by

$$Ay = \Delta y + c^2 y : D(A) = H^2 \bigcap H_0^1 \to L^2.$$

In this case, the degree of unbounded of B is $\gamma = \frac{3}{4} + \epsilon (\epsilon > 0)$ (see [LT00], section 3.1).

We may take $\hat{A}$ as follows

$$\hat{A}h = -\Delta h, \ D(\hat{A}) = H^2 \bigcap H_0^1.$$

Therefore, $(\hat{A}^*)^{-\gamma+\frac{1}{2}} = (\hat{A}^*)^{-\frac{1}{4}+\epsilon} : H^1(\Omega) \to H^{\frac{3}{2}+\epsilon}(\Omega)$.

Remark 4.2 By means of the condition of the degree of unbounded of operator B and (33), we imply that B is admissible.

Indeed, we have

$$\begin{aligned}
\|L_T^*\psi\| &= \sup_{\|u\|_q\le 1}\int_0^T < B^*S^*(T-s)x, u(s) > ds \\
&\le (\int_0^T \|B^*S(t)^*\psi\|^p dt)^{\frac{1}{p}} \\
&\le C_3(\int_0^T \left\|(-\hat{A}^*)^\gamma S(t)^*\psi\right\|^p dt)^{\frac{1}{p}} \\
&\le C_3(\int_0^T \frac{e^{\omega t}}{t^{p\gamma}}\|\psi\|^p dt)^{\frac{1}{p}} dt \ (p\gamma < 1) \\
&\le C_T\|\psi\|.
\end{aligned}$$

Remark 4.3. It is easily seen that assumptions (H_3) (except for the inequalities (35), (36), (42)) and $(H_{4.2})$ hold for most of the classical numerical approximation schemes, such as Galerkin methods, centered finite difference schemes,...Additionally, by using some approximation properties and the properties of the functional $(\hat{A}^*)^{-\gamma+1/2}$, we prove that the inequalities (35), (36), (42) also hold for most of the above classical schemes (see the proof in Section 5). As noted in [LT00], the assumption $H_{4.1}$ of uniform analyticity is not standard, and has to be checked in each specific case.

The main result of the article is the following :

Theorem 3. *Under the previous assumptions, if the control system $\dot{y} = Ay+Bu$ is exactly null controllable in time $T > 0$, then there exist $\beta > 0, h_1 > 0$, and positive real numbers C, C' satisfying*

$$\begin{aligned}
C\left\|e^{TA_h^*}\psi_h\right\|_{X_h}^p &\le \int_0^T \left\|B_h^* e^{tA_h^*}\psi_h\right\|_U^p dt + h^\beta\|\psi_h\|_{X_h}^p \\
&\le C'\|\psi_h\|_{X_h}^p,
\end{aligned} \tag{47}$$

for every $h \in (0, h_1)$ and every $\psi_h \in X_h$, $(1 \le p < 2)$.

In these conditions, for every $y_0 \in X$, and every $h \in (0, h_1)$, there exists a solution $\varphi_h \in X_h$ minimizing the functional

$$J_h(\psi_h) = \frac{1}{p}\int_0^T \left\|B_h^* e^{tA_h^*}\psi_h\right\|_U^p dt + \frac{1}{p}h^\beta\|\psi_h\|_{X_h}^p + < e^{TA_h^*}\psi_h, P_h y_0 >_{X_h}, (1 \le p < 2) \tag{48}$$

and the sequence $(\tilde{Q}_h u_h)_{0<h<h_1}$, where the control u_h is defined by

$$u_h(t) = \left\| B_h^* e^{(T-t)A_h^*} \varphi_h \right\|^{p-2} B_h^* e^{(T-t)A_h^*} \varphi_h,$$

for every $t \in [0,T]$ converges weakly (up to a subsequence), in the space $L^q(0,T;U)$ to a control u such that the solution of :

$$\dot{y} = Ay + Bu, \ y(0) = y_0,$$

satisfies $y(T) = 0$. For every $h \in (0, h_1)$, let $y_h(.)$ denote the solution of

$$\dot{y_h} = A_h y_h + B_h u_h, \ y_h(0) = P_h y_0.$$

Then,

- $y_h(T) = -h^\beta \left\| \varphi_h \right\|^{p-2} \varphi_h$*;*
- *The sequence $(\tilde{P}_h y_h)_{0<h<h_1}$ converges strongly (up to subsequence) in the space $L^q(0,T;X)$, to y(.).*

Futhermore, there exists $M > 0$ such that

$$\int_0^T \|u(t)\|_U^p \leq M^{p/(p-1)} \|y_0|_X^{p/(p-1)},$$

and, for every $h \in (0, h_1)$,

$$\begin{aligned}
&\int_0^T \|u_h(t)\|_{U_h}^p \leq M^{p/(p-1)} \|y_0\|_X^{p/(p-1)}, \\
&h^\beta \|\varphi_h\|_{X_h}^p \leq M^{p/(p-1)} \|y_0\|_X^{p/(p-1)}, \\
&\|y_h(T)\|_{X_h} \leq M^{1/(p-1)} h^{\beta/p} \|y_0\|_X^{1/(p-1)}. \qquad (49)
\end{aligned}$$

Remark 4.4 The left hand side of (47) is uniform observability type inequality for control system (2). This inequality is weaker than the uniform exact null controllability. No attempt has been made here to prove uniform exact null control for the approximation systems (2).

Remark 4.5 A similar result holds if the control system (1) is exactly controllable in time T. However, due to assumption (H_1), the semigroup S(t) enjoys in general regularity properties. Therefore, the solution y(.) of the control system may belong to a subspace of X, whatever the control u is. For instance, in the case of the heat equation with a Dirichlet or Neumann

boundary control, the solution is the smooth function of the state variable x, as soon as $t > 0$,for every control and initial condition $y_0 \in L^2$. Hence, exact controllability does not hold in this case L^2.

Moreover, one may wonder under which assumptions the control u is the control, is defined by (27), such that y(T)=0. As in [LT06], the following proposition give an answer:

Proposition 1. *With the notations of theorem, if the sequence of real numbers $\|\psi_h\|_{X_h}$, $0 < h < h_1$, is moreover bounded, then the control u is the unique control, is defined by (27), such that y(T)=0. Moreover, the sequence $(\tilde{Q}_h u_h)_{0<h<h_1}$ converges strongly (up to a sequence) in the space $L^q(0, T; U)$ to the control u.*

A sufficient condition on $y_0 \in X$, ensuring the boundedness of the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$, is the following : there exists $\eta > 0$ such that the control system $\dot{y} = Ay + Bu$ is exactly null controllable in time t, for every $t \in [T - \eta, T + \eta]$, and the trajectory $t \mapsto S(t)y_0$ in X, for $t \in [T - \eta, T + \eta]$, is not contained in a hyperplane of X.

Other sufficient condition on control u, also ensuring the boundedness of the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$, is the following : there exists $\eta > 0$ such that the control system $\dot{y} = Ay + Bu$ is exactly null controllable in time t, for every $t \in [T - \eta, T + \eta]$, and with the control u is defined as (27), the trajectory $t \mapsto S(t-\xi)Bu(\xi)$ in X, for $t \in [T - \eta, T + \eta]$, every $\xi \in (0, t)$ is not contained in a hyperplane of X.

5 Proof of the main results

1. **The proof of theorem**:

   Proof. For convenience, we first state the following useful approximation lemma, whose proof readily follows that of [LT06], [LT00]. The proof of this lemma is provided in the Appendix.

   Lemma 3. *There exists $C_9 > 0$ such that, for all $t \in (0, T]$ and $h \in (0, h_0)$, there holds*

$$\left\|(e^{tA_h^*}P_h - P_h S(t)^*)\psi\right\|_{X_h} \leq C_9 \frac{h^s}{t} \|\psi\|_X, \tag{50}$$

$$\left\|\tilde{Q}_h B_h^* e^{tA_h^*}\psi_h\right\|_U \leq \frac{C_9}{t^\gamma} \|\psi_h\|_{X_h}, \tag{51}$$

for every $\theta \in [0,1]$.

$$\left\|\tilde{Q}_h B_h^* e^{tA_h^*}\psi_h - B^* S(t)^* \tilde{P}_h \psi_h\right\|_U \leq C_9 \frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma}} \|\psi_h\|_{X_h} \text{ every } \psi_h \in X_h. \tag{52}$$

We carry out proving the theorem as follows:

The degree of unboundedness γ of the control operator B is lower than $\frac{1}{p}$, there exists $\theta \in (0,1)$ such that $0 < \theta + (1-\theta)\gamma < \frac{1}{p}$.

For all $h \in (0, h_0)$ and $\psi_h \in X_h$ we have

$$\begin{aligned}\int_0^T \left\|\tilde{Q}_h B_h^* e^{tA_h^*}\psi_h\right\|_U^p dt &= \int_0^T (\left\|\tilde{Q}_h B_h^* e^{tA_h^*}\psi_h\right\|_U^p - \left\|B^* S(t)^* \tilde{P}_h \psi_h\right\|_U^p) dt \\ &+ \int_0^T \left\|B^* S(t)^* \tilde{P}_h \psi_h\right\|_U^p dt. \end{aligned} \tag{53}$$

We estimate two terms of right hand side of (53).

The control system is exactly null controllable in time T, then there exists a positive real number $C > 0$ such that

$$\int_0^T \left\|B^* S(t)^* \tilde{P}_h \psi_h\right\|^p dt \geq C \left\|S(T)^* \tilde{P}_h \psi_h\right\|_X^p . \tag{54}$$

We have the following inequality

$$|y^p - z^p| < p(y^{p-1} + z^{p-1})\,|y - z|\,, \tag{55}$$

where $y, z \in R^+, p > 1$.

Indeed, we apply mean-value theorem for $f(x) = x^p (p > 1, x \in R^+)$, there exists $\xi \in (y, z)$ such that

$$\begin{aligned}|y^p - z^p| &= |f'(\xi)|\,|y - z| \\ &= p\left|\xi^{(p-1)}\right| . |y - z| \\ &< p(y^{p-1} + z^{p-1}). |y - z|\,.\end{aligned}$$

We apply the above inequality and make use of (40), (28), (44), (50)

to obtain

$$
\begin{aligned}
& \left| \left\| P_h S(T)^* \tilde{P}_h \psi_h \right\|_{X_h}^p - \left\| e^{TA_h^*} \psi_h \right\|_{X_h}^p \right| \\
\leq \;& p(\left\| P_h S(T)^* \tilde{P}_h \psi_h \right\|_{X_h}^{p-1} + \left\| e^{TA_h^*} \psi_h \right\|_{X_h}^{p-1}) \\
& \qquad\qquad\qquad \times \left| \left\| P_h S(T)^* \tilde{P}_h \psi_h \right\|_{X_h} - \left\| e^{TA_h^*} \psi_h \right\|_{X_h} \right| \\
\leq \;& p(C_5 C_1 e^{\omega t} \left\| \psi_h \right\|_{X_h}^{p-1} + C_7 \left\| \psi_h \right\|_{X_h}^{p-1}) . \left\| P_h S(T)^* \tilde{P}_h \psi_h - e^{TA_h^*} \psi_h \right\|_{X_h} \\
\leq \;& C_p \left\| \psi_h \right\|_{X_h}^{p-1} C_9 C_5 h^s \left\| \psi_h \right\|_{X_h} \\
\leq \;& C_{14} h^s \left\| \psi_h \right\|_{X_h}^p .
\end{aligned}
$$

Therefore , from above estimate and (39), we get

$$
\left\| e^{TA_h^*} \psi_h \right\|_{X_h}^p - C_{14} h^s \left\| \psi_h \right\|_{X_h}^p \leq \left\| P_h S(T)^* \tilde{P}_h \psi_h \right\|_{X_h}^p \leq C_5^p \left\| S(T)^* \tilde{P}_h \psi_h \right\|_X^p . \tag{56}
$$

Combine (54) with (56) we have:

$$
\int_0^T \left\| B^* S(t)^* \tilde{P}_h \psi_h \right\|_U^p dt \geq C_{15} \left\| e^{TA_h^*} \psi_h \right\|_{X_h}^p - C_{14} h^s \left\| \psi_h \right\|_{X_h}^p . \tag{57}
$$

For the first term on the right hand side of (53), one has, using (33), (51), (52) and applying the inequality (55)

$$
\begin{aligned}
& \left| \left\| \tilde{Q}_h B_h^* e^{tA_h^*} \psi_h \right\|_U^p - \left\| B^* S(t)^* \tilde{P}_h \psi_h \right\|_U^p \right| \\
\leq \;& p(\left\| \tilde{Q}_h B_h^* e^{tA_h^*} \psi_h \right\|_U^{p-1} + \left\| B^* S(t)^* \tilde{P}_h \psi_h \right\|_U^{p-1}) \\
& \qquad\qquad\qquad \times \left| \left\| \tilde{Q}_h B_h^* e^{tA_h^*} \psi_h \right\|_U - \left\| B^* S(t)^* \tilde{P}_h \psi_h \right\|_U \right| \\
\leq \;& p(\frac{C_9^{p-1}}{t^{\gamma(p-1)}} \left\| \psi_h \right\|_{X_h}^{p-1} + C_3^{p-1} \frac{e^{\omega t(p-1)}}{t^{\gamma(p-1)}} \left\| \psi_h \right\|_{X_h}^{p-1}) \\
& \qquad\qquad\qquad \times \left| \left\| \tilde{Q}_h B_h^* e^{tA_h^*} \psi_h - B^* S(t)^* \tilde{P}_h \psi_h \right\|_U \right| \\
\leq \;& \frac{C_{16}}{t^{\gamma(p-1)}} \left\| \psi_h \right\|_{X_h}^{p-1} . C_9 \frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma}} \left\| \psi_h \right\|_{X_h} \\
\leq \;& C_{17} \frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma+\gamma(p-1)}} \left\| \psi_h \right\|_{X_h}^p .
\end{aligned}
$$

We have $\gamma < \frac{1}{p}(p \geq 1)$, therefore $\theta + (1-\theta)\gamma + \gamma(p-1) < 1$ and we can get, by integration,
$\left| \int_0^T (\left\| \tilde{Q}_h B_h^* e^{tA_h^*} \tilde{P}_h \psi_h \right\|_U^p - \|B^* S(t)^* \psi_h\|_U^p) dt \right| \leq C_{18} h^{s(1-\gamma)\theta} \|\psi_h\|_{X_h}^p$.
Therefore,

$$\int_0^T \left\| \tilde{Q}_h B_h^* e^{tA_h^*} \psi_h \right\|_U^p dt \geq \int_0^T \|B^* S(t)^* \psi_h\|_U^p dt - C_{18} h^{s(1-\gamma)\theta} \|\psi_h\|_{X_h}^p. \tag{58}$$

We choose a real number β such that $0 \leq \beta \leq s(1-\gamma)\theta$. Combine results (53), (57), (58) we have inequality (47).

For $h \in (0, h_1)$, the functional J_h is convex, and inequality (47), is coercive. Therefore, it admits a solution minimum at $\varphi_h \in X_h$ so that

$$0 = \bigtriangledown J_h(\varphi_h) = G_h(T)\varphi_h + h^\beta \|\varphi_h\|^{p-2} \varphi_h + e^{TA_h} P_h y_0,$$

where $G_h(T) = \int_0^T \left\| B_h^* e^{tA_h^*} \varphi_h \right\|^{p-2} e^{tA_h} B_h B_h^* e^{tA_h^*} dt$ is the Gramian of the semidiscrete system.
With $u_h(t) = \left\| B_h^* e^{(T-t)A_h^*} \varphi_h \right\|^{p-2} B_h^* e^{(T-t)A_h^*} \varphi_h$ is chosen then, the solution $y_h(.)$ satisfies

$$\begin{aligned} y_h(T) &= e^{TA_h} y_h(0) + \int_0^T e^{(T-t)A_h} B_h u_h(t) dt \\ &= e^{TA_h} P_h y_0 + G_h(T)\varphi_h \\ &= -h^\beta \|\varphi_h\|^{p-2} \varphi_h. \end{aligned}$$

Note that, since $J_h(0) = 0$, there must hold, at the minimum, $J_h(\varphi_h) \leq 0$. Hence, using the observability inequality (47) and the Cauchy-Schwarz inequality, one gets

$$\begin{aligned} c \left\| e^{TA_h^*} \varphi_h \right\|_{X_h}^p &\leq \int_0^T \left\| B_h^* e^{tA_h^*} \varphi_h \right\|_{U_h}^p + h^\beta \left\| \varphi_h \right\|_{X_h}^p \\ &\leq 2 \left\| e^{TA_h^*} \varphi_h \right\|_{X_h} \left\| P_h y_0 \right\|_{X_h}, \end{aligned}$$

and thus,

$$\left\| e^{TA_h^*} \varphi_h \right\|_{X_h} \leq (\frac{2}{c})^{1/(p-1)} (\left\| P_h y_0 \right\|_{X_h})^{1/(p-1)}. \tag{59}$$

As a consequence,

$$\int_0^T \left\|B_h^* e^{tA_h^*}\varphi_h\right\|_{U_h}^p \leq (\frac{2^p}{c})^{1/(p-1)}(\|P_h y_0\|_{X_h}^{p/(p-1)}), \tag{60}$$

and $h^\beta \|\varphi_h\|_{X_h}^p \leq (\frac{2^p}{c})^{1/(p-1)}(\|P_h y_0\|_{X_h}^{p/(p-1)})$, and the estimates (49) follow. ☐

2. **Proof of proposition**

Proof. If the sequence $(\left\|\tilde{P}_h\varphi_h\right\|_X)_{0<h<h_1}$ is bounded then up to a subsequence, it converges weakly to an element $\varphi \in X$. It follows from the estimate (52) that $u(t) = \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi$ for every $t \in [0,T]$. Moreover, $\tilde{Q}_h u_h$ tends strongly to u in $L^q(0,T;U)$. Indeed, for $t \in [0,T]$,

$$\begin{aligned}
& \tilde{Q}_h u_h(t) - u(t) \\
= \; & \tilde{Q}_h \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} B_h^* e^{(T-t)A_h^*}\varphi_h \\
& \qquad\qquad\qquad - \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi \\
= \; & \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} (\tilde{Q}_h B_h^* e^{(T-t)A_h^*} - B^*S(T-t)^*\tilde{P}_h)\varphi_h \\
& + \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} B^*S(T-t)^*(\tilde{P}_h\varphi_h - \varphi) \\
& + B^*S(T-t)^*\varphi(\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} - \|B^*S(T-t)^*\varphi\|^{p-2}).
\end{aligned} \tag{61}$$

Since the φ_h are bounded, then the $\|u_h\|$ are bounded. From that, we imply the $\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2}$ are bounded .

Using (52), the first term of right hand side of (61) tends to zero clearly. For the second term, for every $t \in [0,T]$ the operator $B^*S(T-t)^*$ is compact, as a strongly limit of finite rank operators and since $\tilde{P}_h\varphi_h - \varphi$ tends to weakly to zero, it follows the second term of the right hand side (61) tends to zero. Furthermore, by applying inequality (55) we get

$$\begin{aligned}
& \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} - \|B^*S(T-t)^*\varphi\|^{p-2} \\
< \; & (2-p)(\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-3} + \|B^*S(T-t)^*\varphi\|^{p-3}) \\
& \qquad \times(\left\|B_h^* e^{(T-t)A_h^*}\varphi_h - B^*S(T-t)^*\varphi\right\|).
\end{aligned}$$

As the $\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-3}$ are bounded and inequality (52) is used again. Hence, the third term tends to zero clearly.

The control u is such that y(T)=0, hence the vector φ must be solution of $\nabla J^*(\psi) = 0$, where J is defined as in Theorem 2. Since J is convex, φ is the minimum of J^*, that is, u is the control such that $y(T) = 0$.

We next prove, by contradiction, that the sufficient conditions provided in the statement of the proposition implies that the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$ is bounded. As the proof of the first sufficient condition is found in [LT06], we give here the proof only for the second sufficient condition. If the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$ is not bounded, then, up to subsequence, $\tilde{P}_h(\varphi_h/\|\varphi_h\|_{X_h})$ converges weakly to Φ in X, as h tends to 0. For every $t \in [T-\eta, T+\eta]$, the control system is exactly null controllable in time t; and thus, from (60), the sequence $\int_0^t < B_h^* e^{(t-s)A_h^*}\varphi_h, Q_h u(s) >_{U_h} ds$ is bounded, uniformly for $h \in (0, h_1)$. Thus, passing to the limit, one gets

$$\int_0^t < \Phi, S(t-s)Bu(s) >_X ds = 0.$$

This equality is equivalent to the fact that : there exists $\xi \in (0,t)$ such that $< \Phi, S(t-\xi)Bu(\xi) >_X = 0$. This contradicts the fact that the trajectory $t \mapsto S(t-\xi)Bu(\xi)$, $t \in [T-\eta, T+\eta]$ and every $\xi \in (0,t)$, is not contained in a hyperplane of X. $\square$

6 Numerical simulation for the heat equation with Dirichlet boundary control

In this section, we give an example of a situation where the theorem 3.1 are satisfied.

Let $\Omega \subset R^n$ be an open bounded domain with sufficiently smooth boundary Γ. We consider the Dirichlet mixed problem for the heat equation:

$$\dot{y} = \Delta y + c^2 y \text{ in } (0,T)\times\Omega$$

$$y(0,.) = y_0 \text{ in } \Omega$$

$$y = u \text{ in } (0,T)\times\Gamma = \Sigma,$$

with boundary control $u \in L^6(0,T; L^2(\Gamma))$ and $y_0 \in L^2(\Omega)$.

Set $X = L^2(\Omega)$ and $U = L^2(\Gamma)$. We introduce the self-adjoint operator:

$$Ah = \Delta h + c^2 h : D(A) = H^2 \cap H_0^1 \to L^2(\Omega).$$

The adjoint $B^* \in L(D(A^*), U)$ of B is given by

$$B^*\psi = -\frac{\partial \psi}{\partial \nu}, \psi \in D(A^*).$$

Moreover, the degree of unboundedness of B is $\gamma = \frac{3}{4} + \epsilon$ ($\epsilon > 0$) (see [LT00], section 3.1).

1. **One-dimensional Finite-Difference semi-discretized model:**

   We next introduce a semi-discretized model of the above heat equation, using 1D Finite-Difference.

   For simplicity, we set $\Omega = (0,1)$, $\Gamma = \{0,1\}$, c=1 and T=1.

   Given $n \in \mathbb{N}$ we define $h = \frac{1}{n+1} > 0$. We consider the following simplex mesh:

   $$\Omega_h = \{x_0 = 0; x_i = ih, i = 1, ..., N; x_{n+1} = 1\},$$

   which divides [0,1] into n+1 subintervals $I_j = [x_j, x_{j+1}]$ j=0,...,n+1. Set

   $$X_h = \{y \in C^0(\Omega_h)\},$$

   $$U_h = \{y \in C^0(\Gamma)\}.$$

   Define $\tilde{P}_h$ (resp., $\tilde{Q}_h$) as the canonical embedding from X_h into $D((-A)^{1/2})$ (resp., from U_h to U). For $x_h \in X_h$ and $u_h \in U_h$, set, $\tilde{P}_h(x_h) = x_h$ and $\tilde{Q}_h(u_h) = u_h$. For $y \in D((-A)^{1/2})' = H^1(\Omega)'$, set $P_h y = (y_1, .., y_i, .., y_{n+2})$ where $y_i = y((i-1)h)$ and, for $u \in U$, set, $Q_h u = (u_1, .., u_i, .., u_{n+2})$ where $u_i = u((i-1)h)$.

   It is clear that the assumptions $(H_{3.1})$ and $(H_{3.3})$ are here satisfied. Our aim is next to verify the inequalities in $(H_{3.2})$ and $(H_{3.4})$.

   In order to get these inequalities, it will necessary to making use of the following usual approximation properties (see [LT00], section 5):

   (i) $\|\Pi_h y - y\|_{H^l(\Omega)} \le ch^{s-l} \|y\|_{H^s(\Omega)}$, $s \le r+1, \quad s-l \ge 0, \quad 0 \le l \le 1$, and the inverse approximation properties

   (ii) $\|y_h\|_{H^\alpha(\Omega)} \le ch^{-\alpha} \|y_h\|_L^2 (\Omega)$, $0 \le \alpha \le 1$.

   (iii) $h^{-1} \|y - \Pi_h y\|_{L^2_{(\Gamma)}} + \left\|(I - \Pi_h)\frac{\partial y}{\partial \nu}\right\|_{L^2(\Gamma)} \le ch^{s-\frac{3}{2}} \|y\|_{H^s(\Omega)}$, $\frac{3}{2} < s < r+1, y \in H^s(\Omega)$.

(iv) $\|y_h\|_{L^2(\Gamma)} + h \left\|\frac{\partial y_h}{\partial \nu}\right\|_{L^2(\Gamma)} \leq Ch^{-\frac{1}{2}} \|y_h\|_{L^2(\Omega)}$, $y_h \in V_h$.
where r is the order of approximation (degree of polynomials) and Π_h is the orthogonal projection of $L^2(\Omega)$ onto V_h .

First, by applying (i) we easily get the inequality (36)

$$\begin{aligned}
\left\|(I - \hat{A^*}^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h)\psi\right\|_{L^2(\Omega)} &\leq Ch^2 \|\psi\|_{H^2(\Omega)} \\
&\leq Ch^2 \|\psi\|_{D(A^*)} \\
&\leq Ch^2 \|A^*\psi\|_X ,
\end{aligned}$$

in this case s=2.

We next verify the inequality (37) as follows

$$\begin{aligned}
&\left\|((-\hat{A}^*)^\gamma)\left(I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h\right)\psi\right\|_X \\
\leq\ & C \left\|(I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h)\psi\right\|_{D((-\hat{A}^*)^\gamma)} \\
\leq\ & Ch^{s-l} \|\psi\|_{D(A^*)} \\
\leq\ & Ch^{s(1-\gamma)} \|A^*\psi\|_X ,
\end{aligned}$$

where we have used (i) with s=2, $D(A^*) = H^s(\Omega)$ and $D((-\hat{A}^*)^\gamma) = H^l(\Omega)$.

For the inequality (39), we apply (iii) with s=2 as

$$\begin{aligned}
\left\|(I - \tilde{Q}_h Q_h)B^*\psi\right\|_{L^2(\Gamma)} &= \left\|(I - \tilde{Q}_h Q_h)\frac{\partial \psi}{\partial \nu}\right\|_{L^2(\Gamma)} \\
&\leq Ch^{1/2} \|\psi\|_{H^2(\Omega)} \\
&\leq Ch^{s(1-\gamma)} \|\psi\|_{D(A^*)} \\
&\leq Ch^{s(1-\gamma)} \|A^*\psi\|_X .
\end{aligned}$$

For the inequality (43), by using (iv) and (40) we get

$$\begin{aligned}
\left\| B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h \psi_h \right\|_U &= \left\| \frac{\partial((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h \psi_h)}{\partial \nu} \right\|_{L^2(\Gamma)} \\
&\leq Ch^{-\frac{3}{2}} \left\| (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h \psi_h \right\|_{L^2(\Omega)} \\
&\leq Ch^{-\frac{3}{2}} \left\| \psi_h \right\|_{X_h}.
\end{aligned}$$

Therefore, the inequality (43) is satisfied for s=2, $\gamma = \frac{3}{4} + \epsilon$.

Moreover, the assumption $(H_{4.2})$ is satisfied with s=2 (see [LT00]). Hence, theorem 3.1 applies, with $\beta = 0.16$, for instance.

We then consider the following finite difference approximation of the above heat equation as follws

$$\begin{aligned}
&\dot{y}_j = \frac{1}{h^2}[y_{j+1} + y_{j-1} - 2y_j] + c^2 y_j \; 0 < t < T, j = 1, ..., n \\
&y_j(0) = y_{j0} \;, j = 1, ..., n \\
&y_0(t) = y_{n+1}(t) = u_h \;, 0 < t < T,
\end{aligned}$$

where $y \in R^{n+2}$, $y_0 \in R^{n+2}$, $u_h \in R$ and

$$\mathbf{A_h} = \frac{1}{h^2} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & (c^2h^2 - 2) & 1 & \dots & 0 \\ 0 & 1 & (c^2h^2 - 2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & (c^2h^2 - 2) & 1 & 0 \\ 0 & \dots & 1 & (c^2h^2 - 2) & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B_h} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

2. **Numerical simulation**

The minimization procedure described in Theorem 3.1 has been implement for d=1, by using a simple gradient method that has the advantages not to require the complex computations and this method can applied with any power p. However, the computation of gradient of J_h is very expensive since the gradient is related to the Gramian matrix.

name	S_h	h	y_0
1D-10	10	10^{-1}	$y_0(x) = e^{-x^2}$
1D-100	100	10^{-2}	$y_0(x) = e^{-x^2}$
1D-500	500	2.10^{-3}	$y_0(x) = e^{-x^2}$

Table 1: Data for the one-dimensional heat equation.

name	$\|\varphi_h\|_X$	$h^\beta(\beta = 0.16)$	$\|y_h(T)\|$
1D-10	0.1690	0.6814	0.4775
1D-100	0.7960	0.4779	0.4565
1D-500	2.0570	0.3699	0.4273

Table 2: Numerical results for one dimensional equation for $\beta = 0.16$.

name	$\|\varphi_h\|_X$	$h^\beta(\beta = 2)$	$\|y_h(T)\|$
1D-10	4.4266	10^{-2}	0.0111
1D-100	4.8933	10^{-4}	1.3467e-004
1D-500	5.0956	4.10^{-6}	5.5178e-006

Table 3: Numerical results for one dimensional equation for $\beta = 2$.

Numerical simulation are carried out with a space-discretization step equal to 0.005, with the data of Table 1. The numerical results are provided in Table 2 for beta =0.16 and Table 3 for beta=2.

The convergence of the method is very slow. From the result of Theorem 3, the final state $y_h(T)$ is equal to $-h^\beta \|\varphi_h\|^{p-2} \varphi_h$ in which φ_h is minimizer of J_h. We note that the maximum value for which the theorem asserts the convergence is very small. For such a small value of β (for instance $\beta = 0.16$), h^β converges to 0 very slowly. It follows that $y_h(T)$ converges very extremely slow.

In practice, the unique minimizer φ_h of J_h is obtained through the simple gradient method in which the step is equal to 0.01 and the error $\epsilon = 10^{-2}$ is taken. With the small value $\beta = 0.16$, it took a long time

to achieve the results in Table 2. Namely, for n=500, it took more one month to get the result after 1000 iterations. It is clearly seen from Table 2 that the convergence of $y_h(T)$ is very slow. These results illustrate the difficult in using our method to compute controls. Although the case beta=2 is not covered by our main theorem, the method seems to converge for this value for beta and we provide hereafter the numerical results in Table 3. Since the value of the beta is greater, the convergence is quicker.

Appendix: proof of lemma

Proof. • First of all, we will prove (51)

For every $\psi \in D(A^*)$, one has

$$\left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* \left(\hat{A}^*\right)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi\right\|_U \leq \left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi\right\|_U + \left\|B^* \left(\hat{A}^*\right)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi\right\|_U . \tag{62}$$

We estimate each term of the right hand side of (62). Since $B_h^* = Q_h B^* \left(\hat{A}^*\right)^{-\gamma+\frac{1}{2}} \tilde{P}_h$ and thus, using (40) (42) (44) one gets

$$\begin{aligned} \left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi\right\|_U &\leq C_5 \left\|B^* \left(\hat{A}^*\right)^{-\gamma+\frac{1}{2}} \tilde{P}_h e^{tA_h^*} P_h \psi\right\|_U \\ &\leq C_5 C_6 h^{-\gamma s} \left\|e^{tA_h^*} P_h \psi\right\|_{X_h} \\ &\leq C_5^2 C_6 C_7 h^{-\gamma s} e^{\omega t} \|\psi\|_X . \end{aligned} \tag{63}$$

On the other hand, from (28), (40), (42) one obtains

$$\begin{aligned} \left\|B^* \left(\hat{A}^*\right)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi\right\|_U &\leq C_6 h^{-\gamma s} \|P_h S(t)^* \psi\|_{X_h} \\ &\leq C_5 C_6 h^{-\gamma s} \|S(t)^* \psi\|_X \\ &\leq C_1 C_5 C_6 h^{-\gamma s} e^{\omega t} \|\psi\|_X . \end{aligned} \tag{64}$$

Hence, combine (63), (64) with (62), there exists $C_{10} > 0$ such that

$$\left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* \left(\hat{A}^*\right)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi\right\|_U \leq C_{10} h^{-\gamma s} \|\psi\|_X . \tag{65}$$

for every $\psi \in D(A^*)$, every $t \in [0,T]$, and every $h \in (0,h_0)$.

Moreover, we get another estimate of this term. By using (33), (38), (40) , (42), (50) one gets

$$
\begin{aligned}
& \left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \\
= & \left\| \tilde{Q}_h Q_h B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \\
\leq & \left\| \tilde{Q}_h Q_h B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h (e^{tA_h^*} P_h \psi - P_h S(t)^* \psi) \right\|_U \\
& + \left\| \tilde{Q}_h Q_h B^* ((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h - I) S(t)^* \psi \right\|_U \\
& + \left\| (\tilde{Q}_h Q_h - I) B^* S(t)^* \psi \right\|_U \\
& + \left\| B^* (I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h) S(t)^* \psi \right\|_U \\
\leq & C_5 C_6 h^{\gamma s} \left\| e^{tA_h^*} P_h \psi - P_h S(t)^* \psi \right\|_{X_h} \\
& + C_5 C_3 \left\| (-\hat{A})^{\gamma} ((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h - I) S(t)^* \psi \right\|_X \\
& + C_4 h^{s(1-\gamma)} \| A^* S(t)^* \psi \|_X \\
& + C_3 \left\| (-\hat{A})^{\gamma} ((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h - I) S(t)^* \psi \right\|_X \\
\leq & C_5 C_6 C_9 \frac{h^{s(1-\gamma)}}{t} \|\psi\|_X + (C_3(C_5+1)+1) C_4 h^{s(1-\gamma)} \| A^* S(t)^* \psi \|_X \\
\leq & C_{11} \frac{h^{s(1-\gamma)}}{t} \|\psi\|_X. \qquad (66)
\end{aligned}
$$

Then, raising (65) to the power $1-\gamma$, (66) to power to γ and multiplying both result estimates, we obtain

$$
\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \leq \frac{C_{12}}{t^{\gamma}} \|\psi\|_X.
$$

Hence,

$$
\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi \right\|_U \leq \frac{C_{12}}{t^{\gamma}} \|\psi\|_X + \left\| B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U. \qquad (67)
$$

From (29), (33), (36) one yieds

$$\begin{aligned}
\left\| B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^*\psi \right\|_U &\leq \left\| B^*(I-(\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h) S(t)^*\psi \right\|_U + \|B^* S(t)^*\psi\|_U \\
&\leq C_{13}\frac{e^{\omega t}}{t^\gamma}\|\psi\|_X. \qquad (68)
\end{aligned}$$

Combine (67) with (68) and by setting $\psi = \tilde{P}_h\psi_h$ we get (51).

- Finally, we prove (52). On the one hand, reasoning as above for obtaining (66), we get

$$\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h\psi - B^* S(t)^*\psi \right\|_U \leq C\frac{h^{s(1-\gamma)}}{t}\|\psi\|_X, \qquad (69)$$

for every $\psi \in D(A^*)$, every $t \in [0,T]$ and every $h \in (0,h_0)$.

On the other hand, from (51) and setting $\psi = \tilde{P}_h\psi_h$ one obtains

$$\begin{aligned}
\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h\psi - B^* S(t)^*\psi \right\|_U &\leq \left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h\psi \right\|_U + \|B^* S(t)^*\psi\|_U \\
&\leq \frac{C_9}{t^\gamma}\|\psi_h\| + C_3\left\| (-\hat{A}^*)^\gamma S(t)^*\psi \right\|_X \\
&\leq \frac{C}{t^\gamma}\|\psi_h\|_X. \qquad (70)
\end{aligned}$$

Raising (69) to the power θ, (70) to the power $1-\theta$ and multiplying both resulting estimates, we obtain (52).

The proof of the inequality (50) is found in [LT06], [LT00].

□

7 Conclusion

We have shown that the appropriate duality techniques can be applied to solve (3), namely the Fenchel-Rockafellar theorem.

Additionally, it is also stated that under standard assumptions on the discretization process, for an exactly null controllable linear control system, if the semigroup of approximating system is uniformly analytic, and if the degree of unboundedness of the control operator is greater than $\frac{1}{2}$ then the unform observability type inequality is proved. Consequently, a minimization procedure was provided to build the aproximation controls. This is implemented in the case of the one dimensional heat equation with Dirichlet boundary control.

Note that, we only stress our problem on the case $\gamma \geq 1/2$. Some relevant problems for which $\gamma < 1/2$ that are refered to [LT06].

One open question is given: how the above results change if we remove the assumption of uniform analyticity of the discretized semigroup.

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References

[BHL10a] Boyer F, Hubert F and J. Le Rousseau , *Discrete Carleman estimates for the elliptic operators and uniform controllability of semi discretized parabolic equations*, J. Math. Pur. Appl., vol 93 (2010), pages 240-276.

[BHL10b] Boyer F, Hubert F and J. Le Rousseau , *Discrete Carleman estimates for the elliptic operators in arbitrary dimension and applications*, SIAM J. Control Optim, vol 48 (2010), pages 5357-5397.

[CGL94] Carthel C, Glowinski R and Lions J L, *On exact and approximate Boundary Controllabilities for the heat equation: a numerical approach*, J. Optimal. Theory Appl., vol 82 (1994), pages 429-484.

[CT06] Y. Chitour, E. Trelat, *Controllability of partial differential equations, Advanced Topics in Control Systems Theory*, Lecture Notes in Control and Inform. Sci, vol 328, Spinger, London (2006).

[ET99] I. Ekeland, R. Temam, *Convex analysic and variational problems*, Classics in Applied Mathematics 28, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1999.

[E] Lawrence C. Evans, *An introduction to Mathematical Optimal Control Theory*, Version 2.

[F87] H. O. Fattorini, *A unified theory of necessary conditions for nonlinear nonconvex control systems*, Appl. Math. Optim. , 15 (1987), pp. 141-185

[FF91] H. O. Fattorini and H. Frankowska, *Necessary conditions for infinite dimensional problems, Mathematics of Control*, Signals, and Systems, (1991), Spinger - Verlag New York.

[CM09] E. Fernandez- Cara and A. Munch, *Numerical null controllability of the 1D heat equation: primal algorithms*. Prepint (2009).

[CM10] E. Fernandez- Cara and A. Munch, *Numerical null controllability of the 1D heat equation: dual algorithms*. Prepint (2010).

[GLH08] R. Glowinski, J.L. Lions and J. He, *Exact and approximate controllability for distributed parameter systems: a numerical approach*, Encyclopedia of Mathematical and its Applications, 117. Cambridge University Press, Cambridge, 2008.

[IZ99] J.A Infante, E. Zuazua, *Boundary observability for the space semi-discretizations of the 1-D wave equation*, M2AN Math. Model. Number. Anal. 33 (2) (1999) 407-438.

[L88] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev. 30 (1988) 1-68.

[LT00] I.Lasiecka, R.Triggiani, *Control theory for partial differential equation: continuous and approximation theories*. I. Abstract parabolic systems, Encyclopedia of Mathematics and its Applications, vol. 74, Cambridge University Press, Cambridge, 2000.

[LT06] S. Labbe , E. Trelat, *Uniform controllability of semidiscrete approximations of parabolic control system*, Systems and Control Letters, vol 55 (2006), pages 597-609.

[LY85] X. Li and Y. L. Yao, *Maximum Principle of distributed parameter systems with time lags*, Lecture Notes in Control and Information Sciences, vol. 75, Spinger- Verlag, New York, 1985, pp. 410-427.

[LY91] X. Li and J. Yong, *Necessary conditions for optimal control of distributed parameter systems*, SIAM J. Control Optim., vol. 29, number 4, 1991, pp. 895-909.

[LZ02] L. Leon. E. Zuazua, *Boundary controllability of the finite-difference space semi-discretizations of the beam equation*, ESAIM Control Optim. Calc. Var. 8 (2002) 827-862.

[MZ10] A. Munch and E. Zuazua, *Numerical approximation of the null controls for the heat equation: ill-posedness and remidies*, Inverse Problems 26 (2010), no.8, 085018, 39 pp.

[LZ98] A. Lopez, E. Zuazua, *Some new results related to the null controllability of the 1-D heat equation*, Sem. EDP, Ecole Polytechnique, VIII, 1998, 1-22.

[NZ03] M. Negreanu, E. Zuazua, *Uniform boundary controllability of discre 1-D wave equation*, Systems Control Lett. 48 (3-4) (2003) 261-279.

[P83] A.Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. 44 (1983).

[PM10] P. Pedregal, A. Munch, *Numerical null controllability of the heat equation through a variational approach*, 2010

[PS62] Pontryagin, L.S. et al. *The Mathematical Theory of Optimal Processes*, vol. 4. Interscience, 1962

[T05] E. Trelat, (French version) Optimal control : Theory and applications, Concrete Mathematics, Vuibert, Paris, 2005, 246 pages.

[TW07] M.Tucsnak, G.Weiss, *Observability and controllability of infinite dimensional systems*, 2007.

[Zua99] E. Zuazua, *Boundary observability for finite-difference space semi-discretizations of the 2-D wave equation in the square*, J. Math. Pures Appl. (9) 78 (1999) 523-563.

[Zua02] E. Zuazua, *Controllability of the partial differential equations and its semi-discrete approximations*, Discrete Contin. Dyn. Syst. 8 (2) (2002) 469-513.

[Zua04] E. Zuazua, *Optimal and approximate control of finite-difference approximation schemes for 1-D wave equation*, Rendiconti di Matematica VIII, 24, II (2004) 201-237.

[Zua05] E. Zuazua, *Propagation, Observation, Control and Numerical Approximation of Wave approximated by finite-difference method*, SIAM Review 47, 2 (2005) 197-243.

[Zua06] E. Zuazua, *Control and numerical approximation of the wave and heat equations*, International Congress of Mathematicians, Madrid, Spain III (2006), 1389-1417.